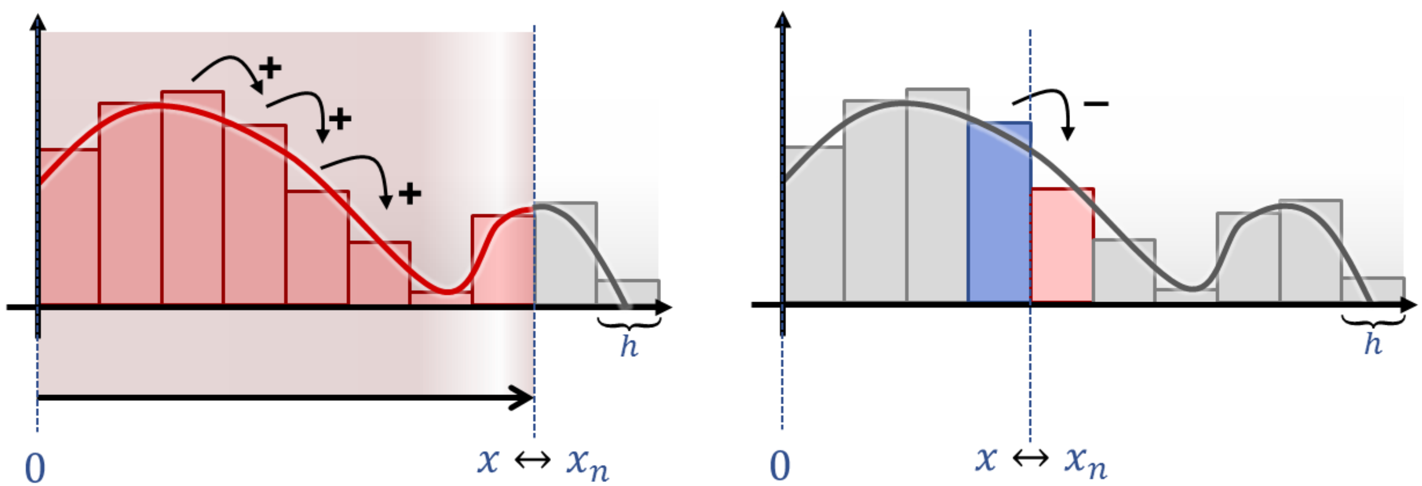
Modelling 1

SUMMER TERM 2020



LECTURE 20

Differential & Functional Equations

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Functional Equations

(usually: Differential & Integral Equations)

Functional Equations

Searching for functions

- Implicit definition of a function

$$F(f) = 0$$

  - Function f: unknown
  - Function F: constraint on f

Functional Equations

Linear functional equations

- F is linear ("linear operator").
  - We use L in that case.
  - We drop the $(\cdot)$ in analogy to matrix multiplication
- Homogeneous:

$$Lf = 0$$

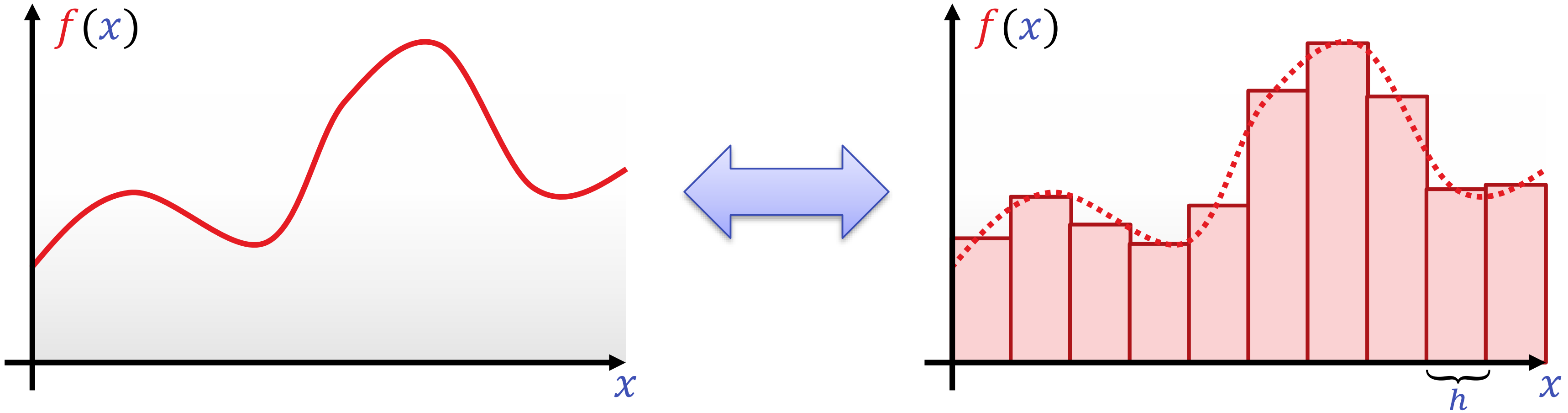
- Inhomogeneous:

$$Lf = g$$

Discrete Analogy

Function f

Think of this:



We Know the Structure...

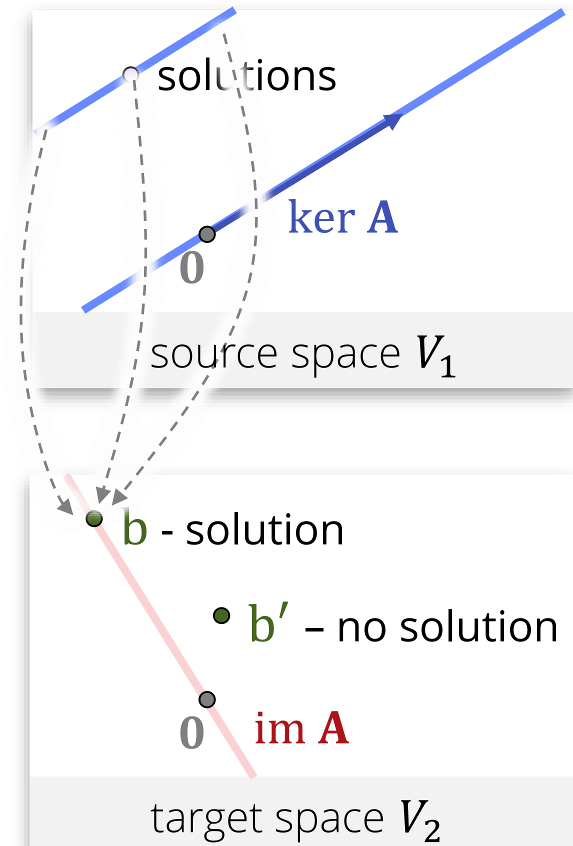
Structure of a functional equation

- Linear solution space
- Add inhomogeneous solution to any homogeneous

Reminder (Linear Maps)

Solutions to linear system

- $\mathbf{A}\mathbf{x} = \mathbf{0}$
  - Solution space = $\ker \mathbf{A}$
- $\mathbf{A}\mathbf{x} = \mathbf{b}$
  - Solution if and only if $\mathbf{b} \in \operatorname{im} \mathbf{A}$
- Set of all solutions:
  - One $\mathbf{y}$ with $\mathbf{A}\mathbf{y} = \mathbf{b}$
  - Add any solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$
  - Solution set: $\mathbf{y} + \ker \mathbf{A}$



Differential Equations

Ordinary

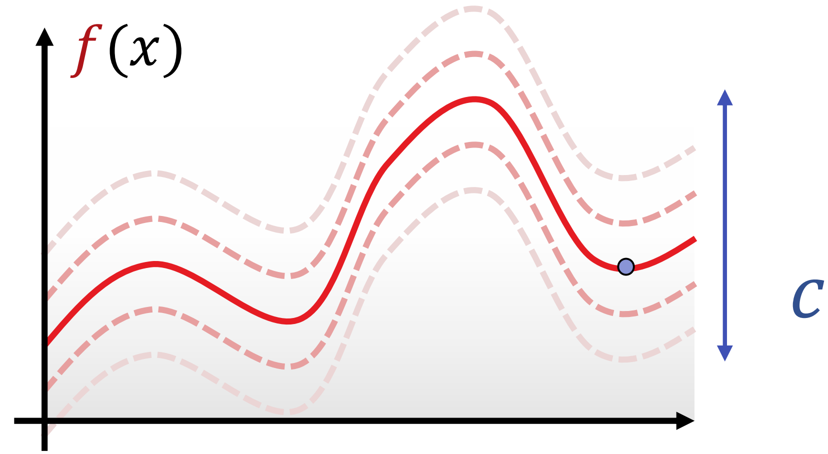
Differential Equations
(ODEs)

Examples

Boundary Conditions

Solution space

- $\frac{d}{dx} f(x) = g(x)$
- $\dim \ker \frac{d}{dx} = 1$
- Solution space is 1-dimensional
- One degree of freedom

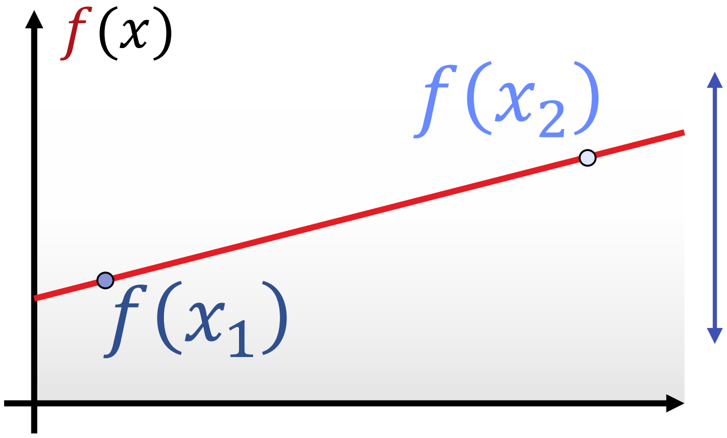


$$f(x) = \int g(x) + c$$

Boundary Conditions

Solution space

- $\frac{d^2}{dx^2} f(x) = 0$
- $\dim \ker \frac{d^2}{dx^2} = 2$
- Solution space is 2-dimensional
- Two degrees of freedom



Discretized

Continuous Equation

$$\frac{d}{dx} f(x) + 1 \cdot f(x) = 0$$

Matrix Approximation

$$\left[\frac{1}{h}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = 0$$

Discretized

Continuous Equation

$$\frac{d^2}{dx^2} f(x) = 0$$

Matrix Approximation

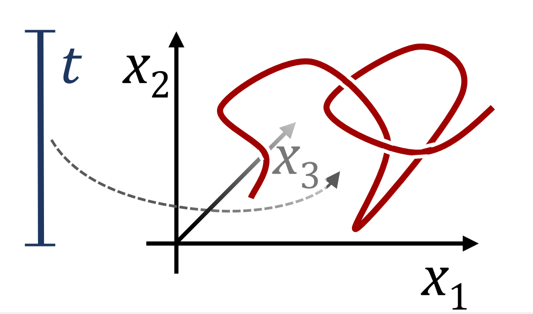
$$\frac{1}{h^2}\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = 0$$

What is it about?

Designing Curves

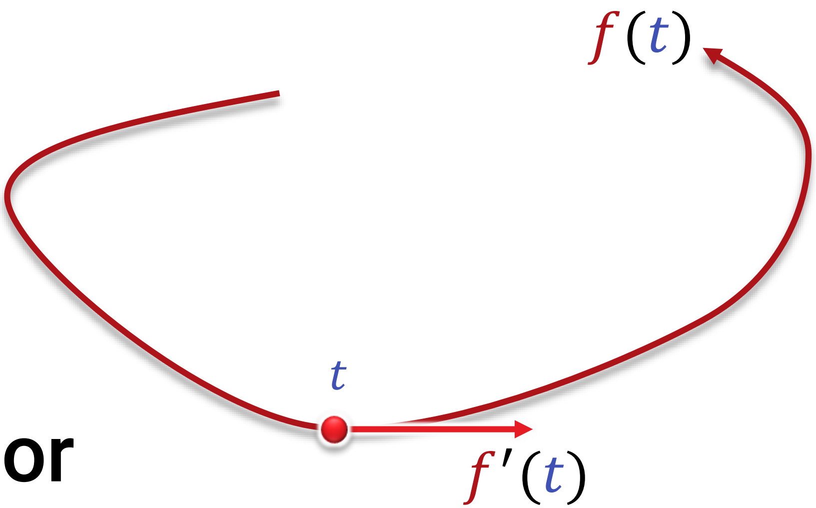
One Parameter functions

- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ("scalar field")
- Functions $f: \mathbb{R} \rightarrow \mathbb{R}^n$ ("curves")
- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (general case)



curves

Geometric Meaning



Tangent Vector

- f' is the tangent vector
  - Higher order derivatives: also vectors
- Physical particle
  - First derivative $\dot{f}$ ≙ velocity.
  - Second derivative $\ddot{f}$ ≙ acceleration

Example

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + cf(t) = g(t)$$

Example

- Linear
- 1dim ODE
- 2nd degree

First Order Derivatives Suffice

Higher order ODE

$$\frac{d^2}{dt^2} f(t) = g(t)$$

Convert to system (multi-dim.) of lower order DEs

Substitution

$$v(t) := \frac{d}{dt} f(t)$$

$$\frac{d^2}{dt^2} f(t) = \frac{d}{dt} v(t)$$

System

$$v(t) = \frac{d}{dt} f(t)$$

$$\frac{d}{dt} v(t) = g(t)$$

General Form of an ODE

Unknown

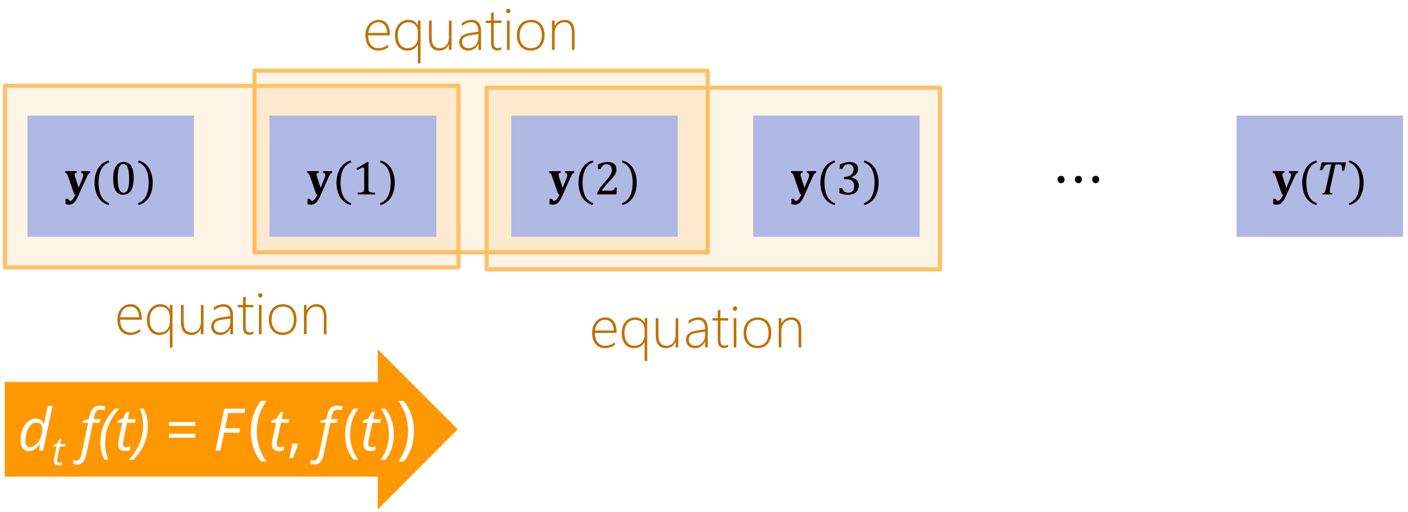
$$f : \mathbb{R} \to \mathbb{R}^n$$

Explicit Form

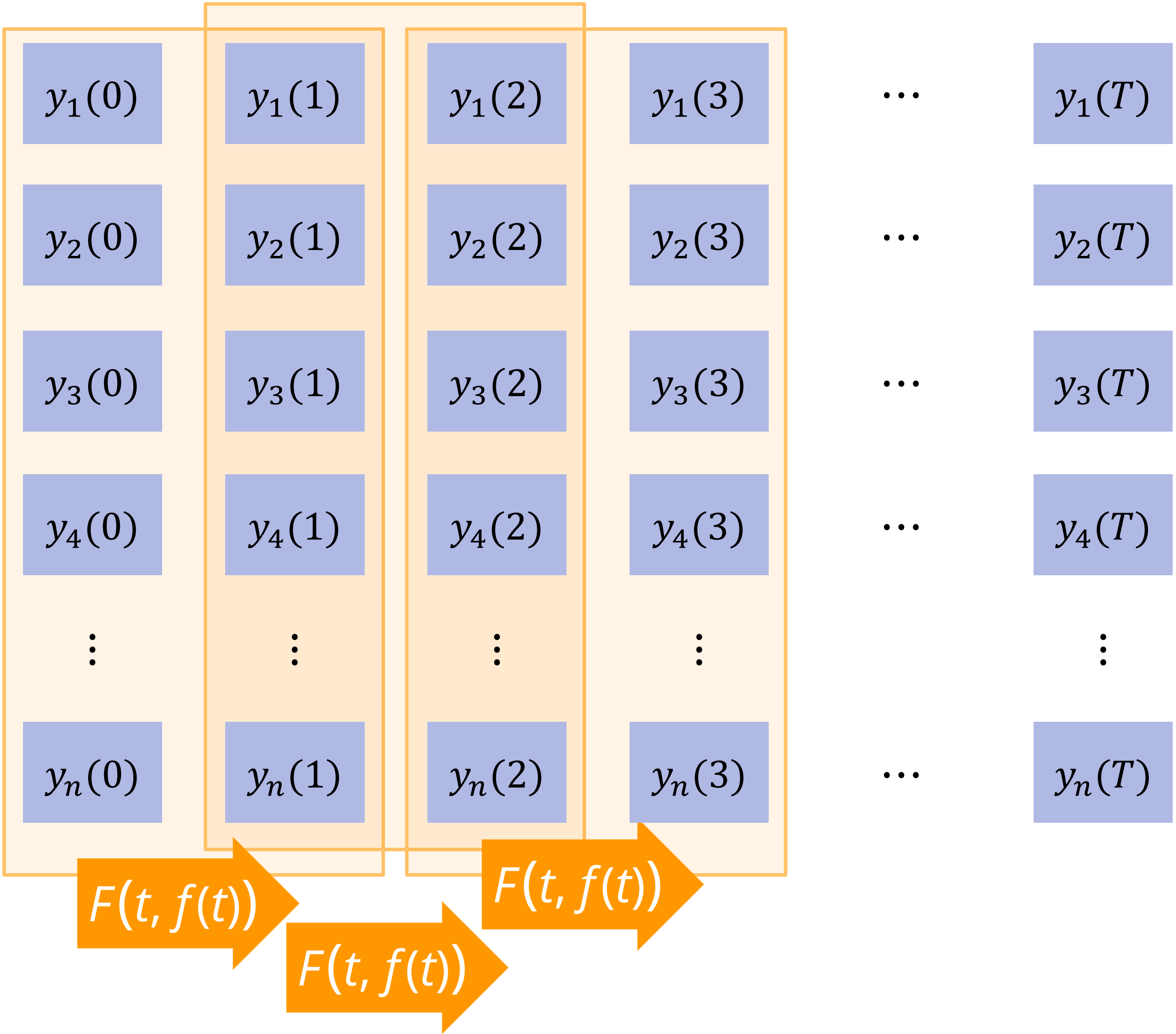
$$\frac{d}{dt} f(t) = F\big(t, f(t)\big)$$

Chain-Structure

Causal Chain

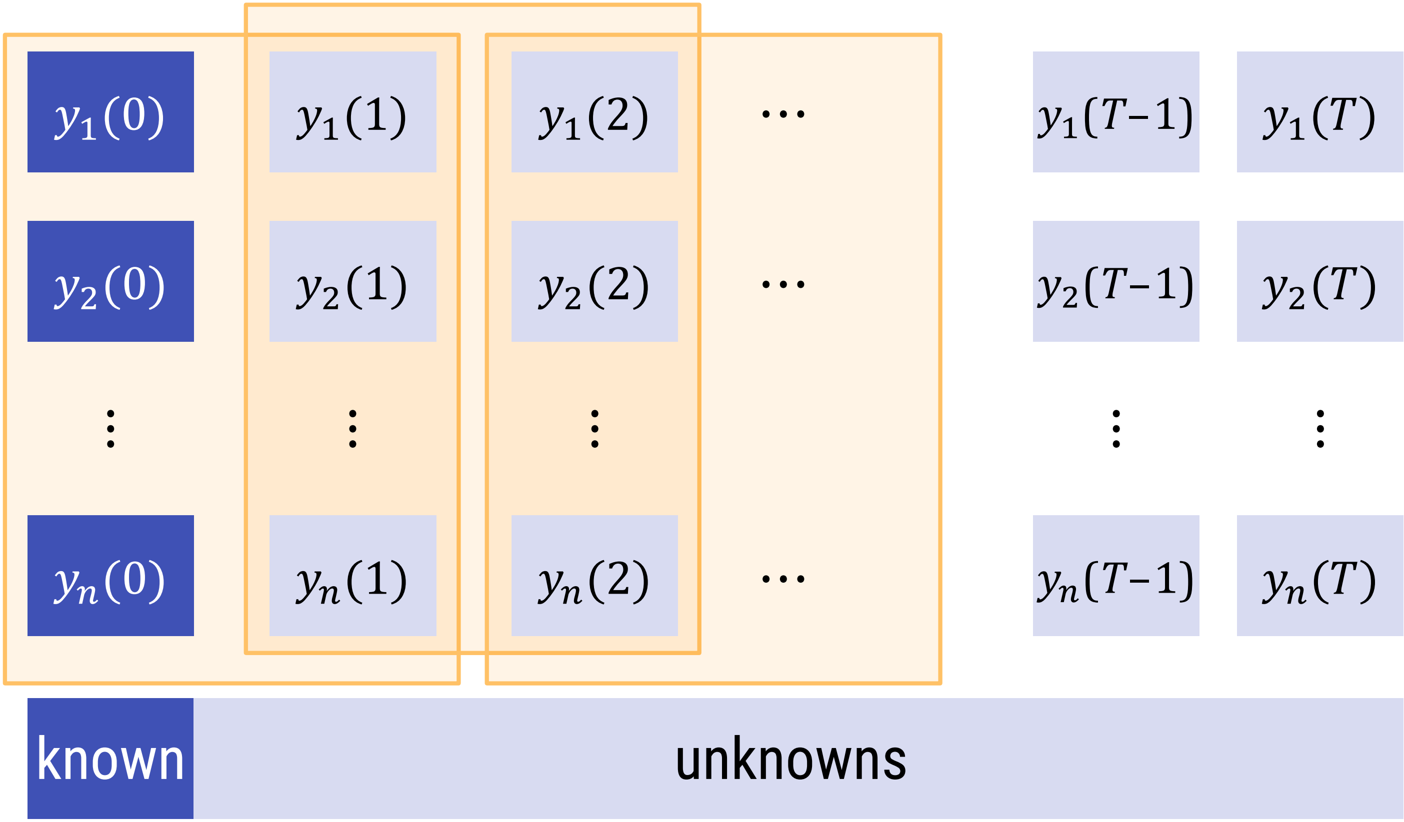


Structure of ODE



Initial Value Problems

Initial Value Problems



Solution

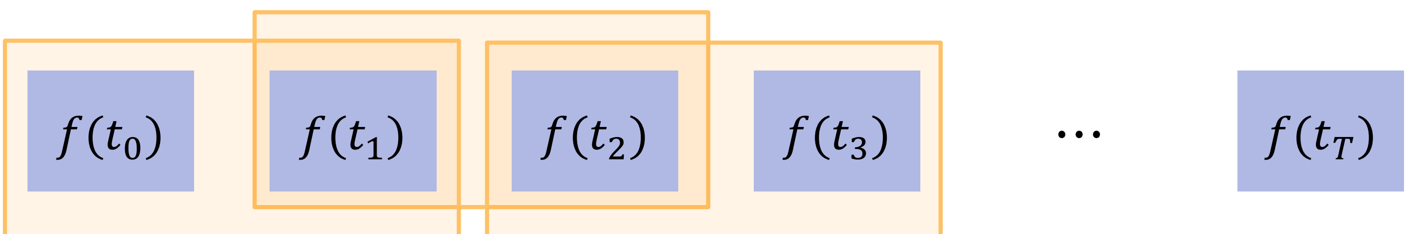
- Solve step-by-step
- Propagate information forward

Numerical Solvers

ODE

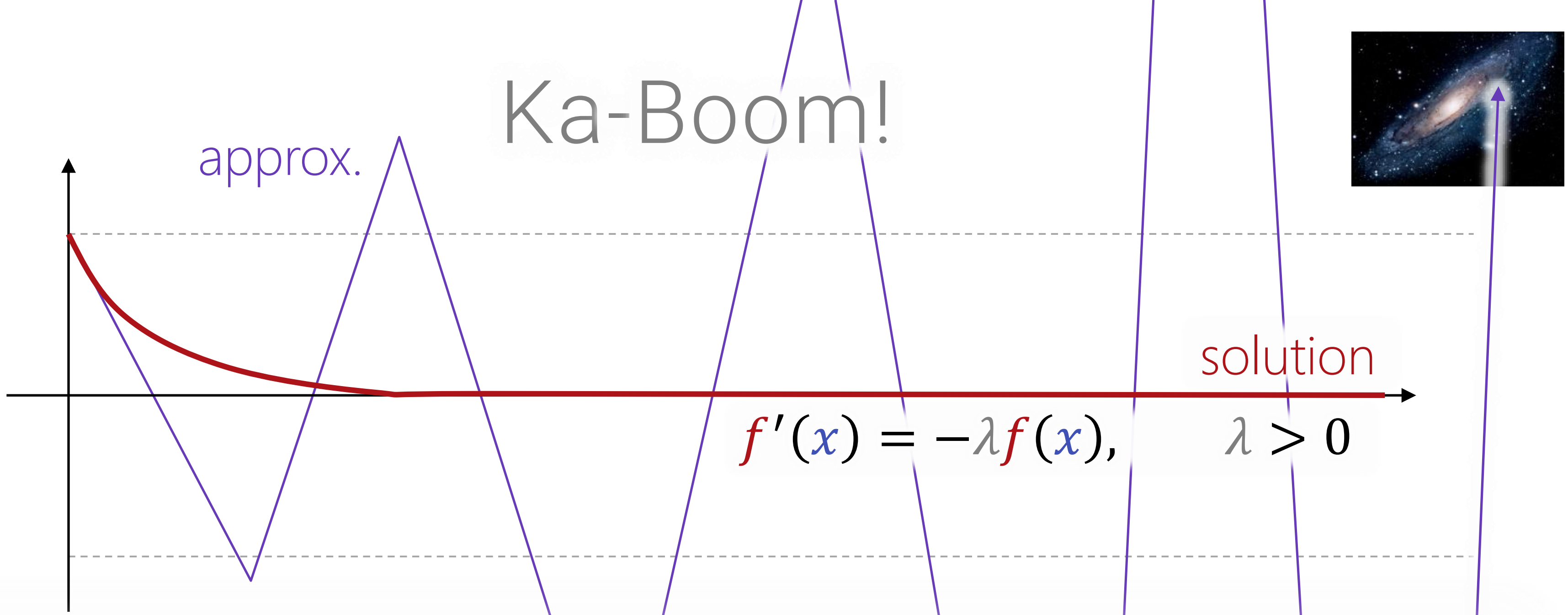
$$f : \mathbb{R} \to \mathbb{R}^n, \qquad \frac{d}{dt} f(t) = F\big(t, f(t)\big)$$

Explicit Euler integrator



$$f(t_{i+1}) = f(t_i) + F\big(t_i, f(t_i)\big) \cdot (t_{i-1} - t_i)$$

Ka-Boom!

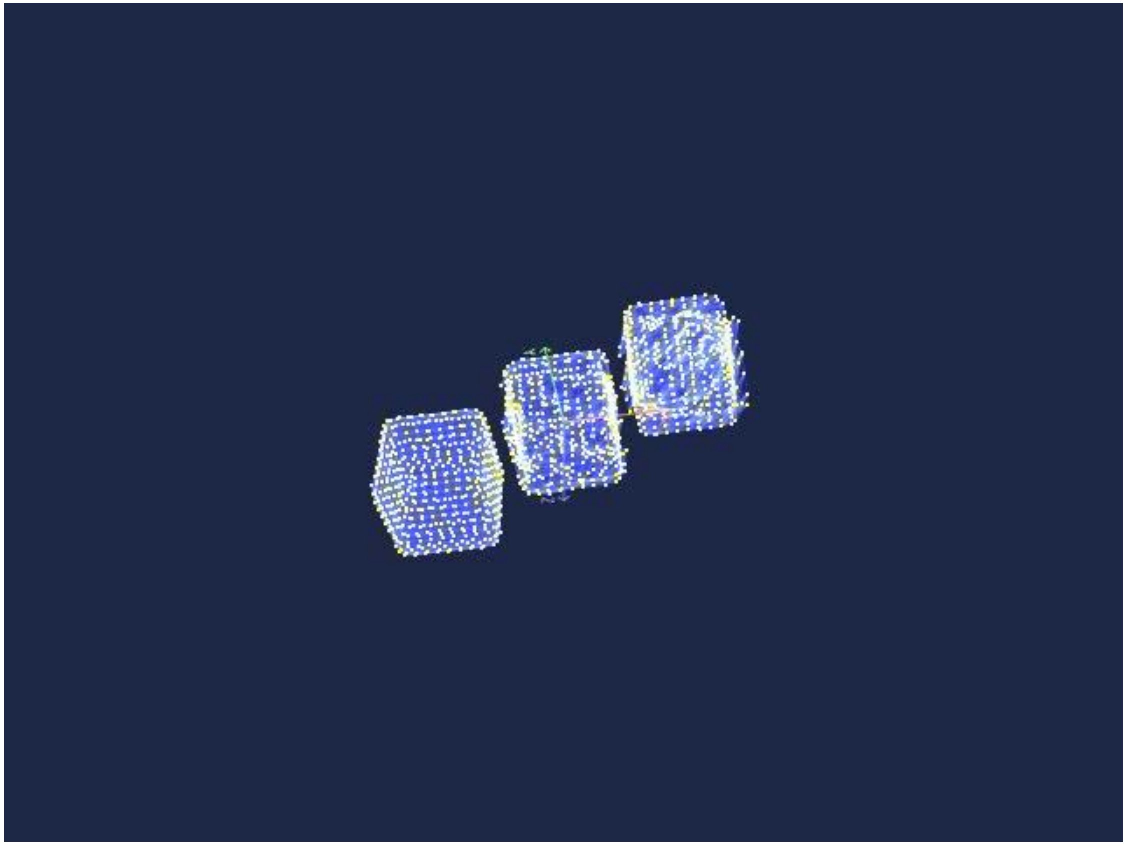


Explicit Euler integrator

$$f(t_{i+1}) = f(t_i) + F\big(t_i, f(t_i)\big) \cdot (t_{i-1} - t_i)$$

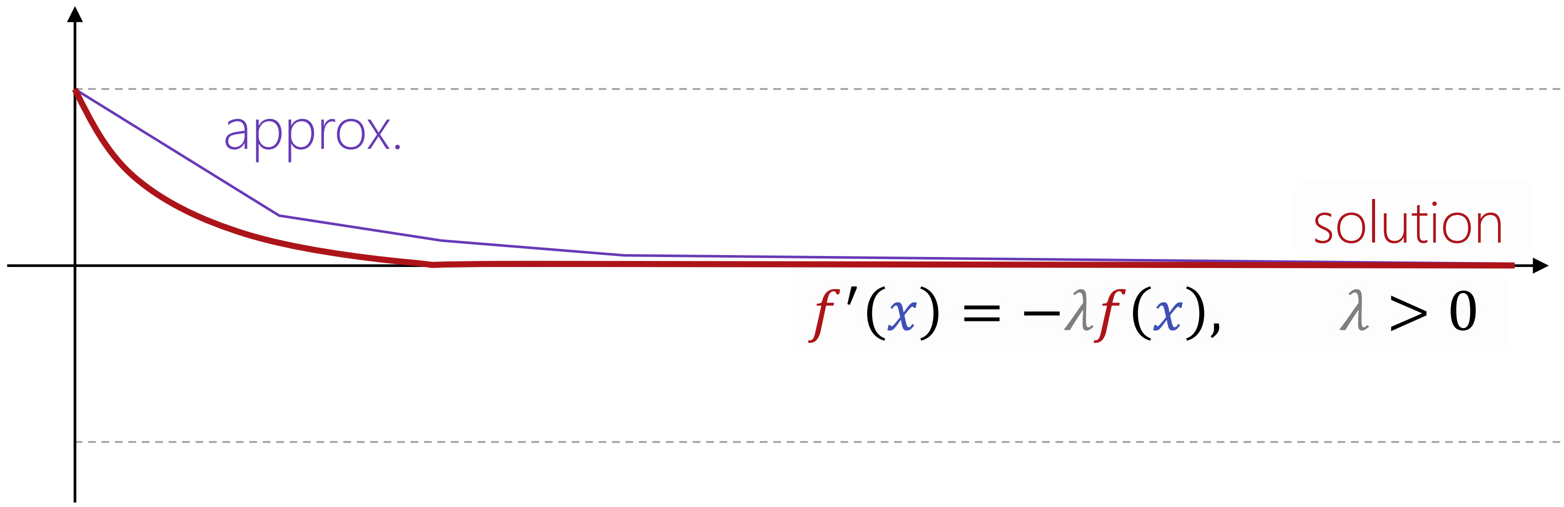
time step must be smaller than $|2\lambda|$, λ being the most negative eigenvalue of F

An Actual Example...



Stiff elasticity model (that I screwed up myself :-))

Ka-Boom!



Implicit Euler integrator

$$f(t_{i+1}) = f(t_i) + F\big(t_{i+1}, f(t_{i+1})\big) \cdot (t_{i-1} - t_i)$$

unconditionally stable
(does not mean accurate)

downside:
need to solve system of equations

Integrators - Variants

Higher consistency order

- Local polynomial approximation

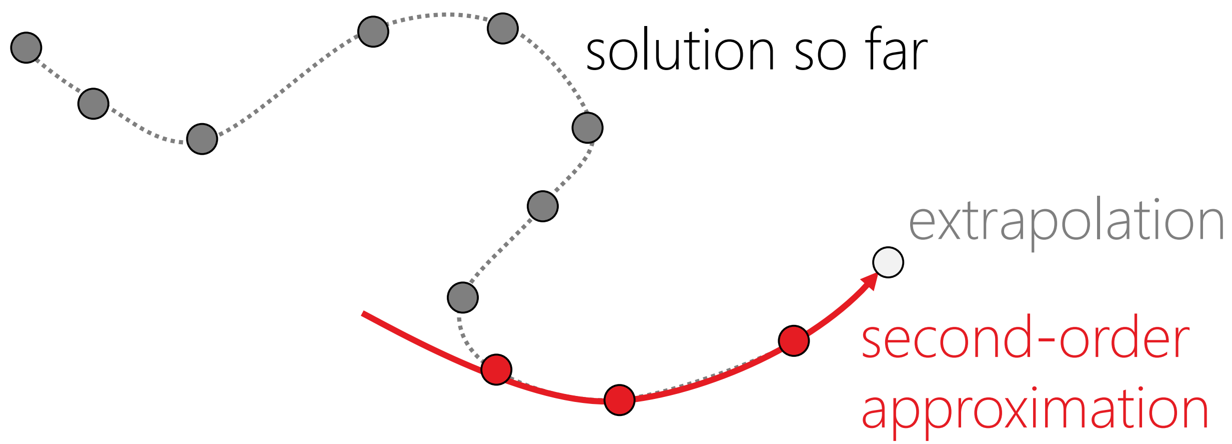
Basic Idea

- Local polynomial approximation
- Fitted to multiple evaluations of $F(t, y)$

Two Variants

- Runge-Kutta Methods: single time step
  - RK4 most popular
- (Linear) Multi-Step Methods: incl. previous time steps
  - BDF-2 popular for stiff problems (i.e., huge $|\lambda_{min}|$)

Integrators - Variants



Multi-Step Methods

- Linear MSM: Fit polynomial to last k steps
  - Explicit: predict next value
  - Implicit: optimize for next value
  - Specific implicit MSM of degree 2 (BDF-2 method) is very stable and accuracy is ok

Analytical Solutions

ODE

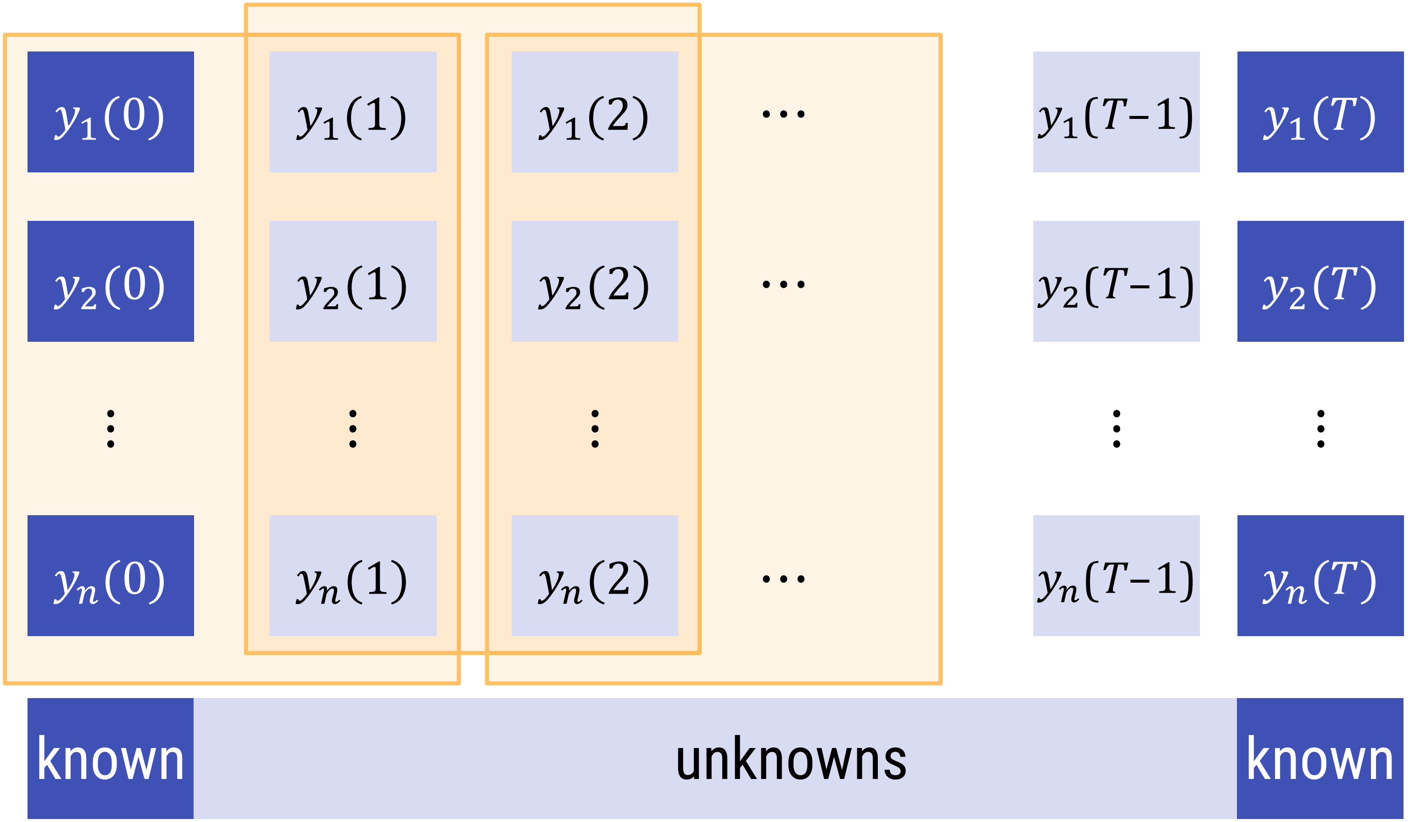
$$f : \mathbb{R} \to \mathbb{R}^n, \qquad \frac{d}{dt} f(t) = F\big(t, f(t)\big)$$

Linear ODEs

- F linear in variables
- Analytical solution possible
  - Via matrix factorization
  - Jordan-normal-form (in $\mathbb{C}$)
  - Solution based on complex exponentials
- Shift-invariant: Fourier transforms

Boundary Value Problems

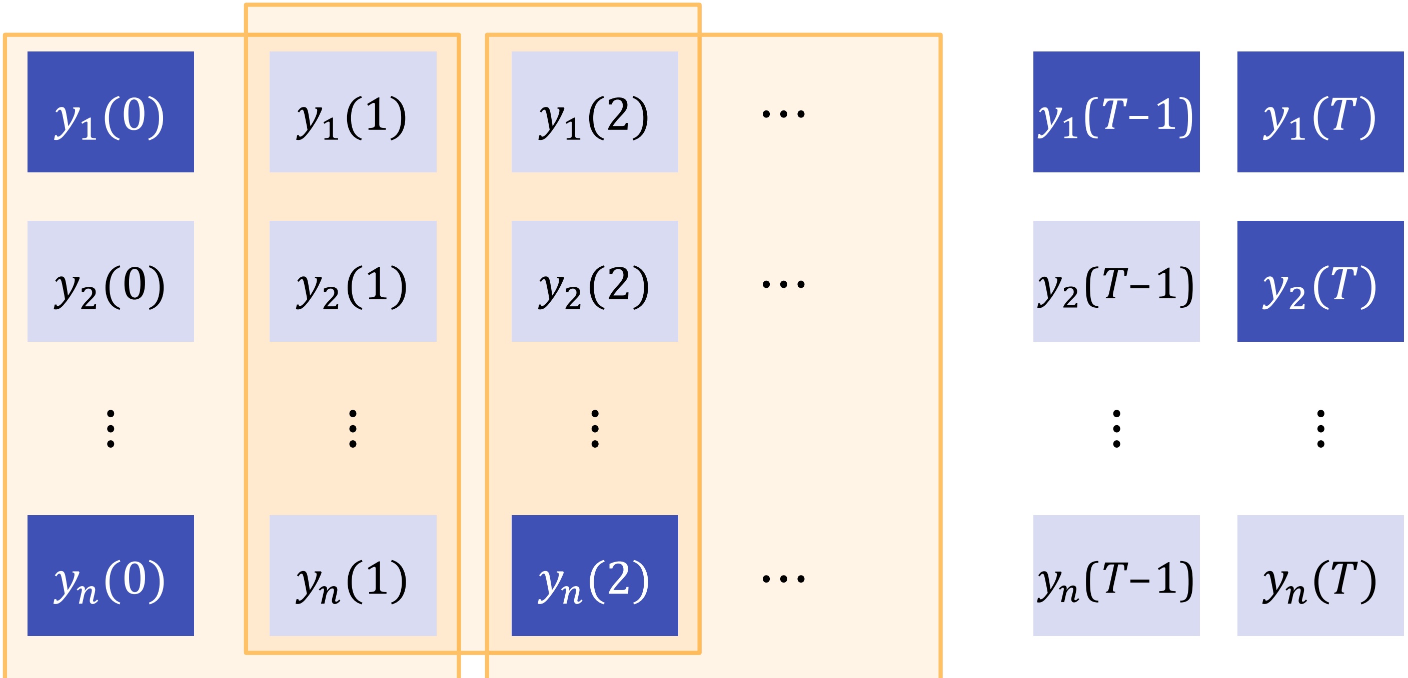
Boundary Value Problems



Solution

- Step-by-step does not work
- Need to solve global system of equations

General constraints



Variant

- Impose constraints all over the place
  - Analog to diffusion Images (link in script)
- Careful with degrees of freedom
  - We rather go for least-squares (more later)

ODE Example:

Newtonian Physics

Example

Newtonian Physics

"$F = m \cdot a$"

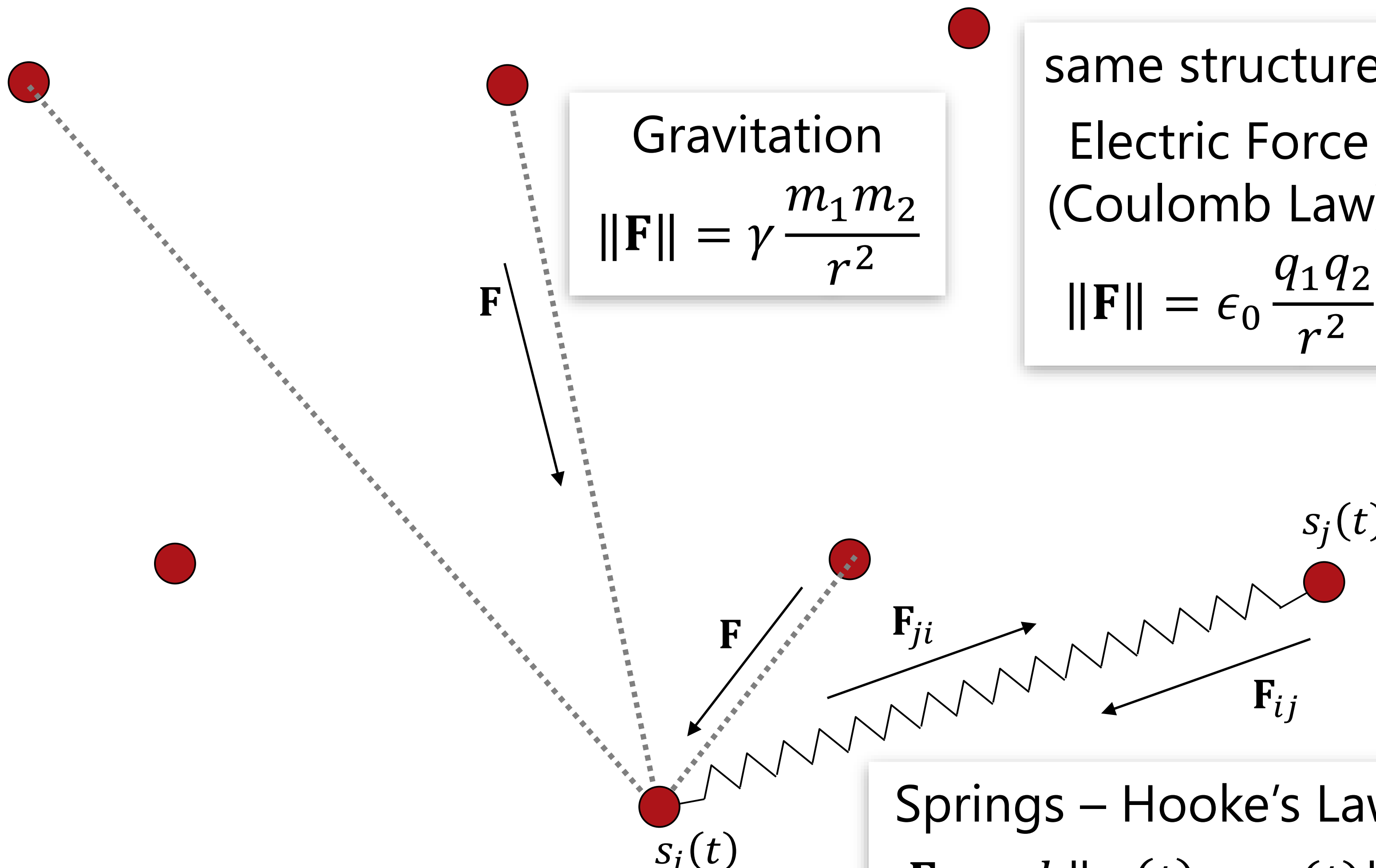
Which means

$$F\big(t, s(t)\big) = m \cdot a(t) = m \cdot \ddot{s}(t)$$

In other words...

$$\frac{d^2}{dt^2} s(t) = \frac{1}{m} F\big(t, s(t)\big)$$

Particle Systems



Gravitation

$$\|\mathbf{F}\| = \gamma \frac{m_1 m_2}{r^2}$$

same structure:

Electric Force (Coulomb Law)

$$\|\mathbf{F}\| = \epsilon_0 \frac{q_1 q_2}{r^2}$$

Springs – Hooke's Law

$$\mathbf{F}_{ij} = k\|s_j(t) - s_i(t)\|$$

Partial

Differential Equations
(PDEs)

Multi-dimensional Inputs

One Parameter functions

- Functions $f: \mathbb{R}^n \to \mathbb{R}$ ("scalar field")
- Functions $f: \mathbb{R} \to \mathbb{R}^n$ ("curves")
- Functions $f: \mathbb{R}^n \to \mathbb{R}^m$ (general case)

We'll stick to that case for simplicity.

No fundamental difference.

Partial Differential Equations

Unknown

$$f : \mathbb{R}^n \to \mathbb{R}$$

Explicit Form

$$Df(\mathbf{x}) = F\big(\mathbf{x}, f(\mathbf{x})\big)$$

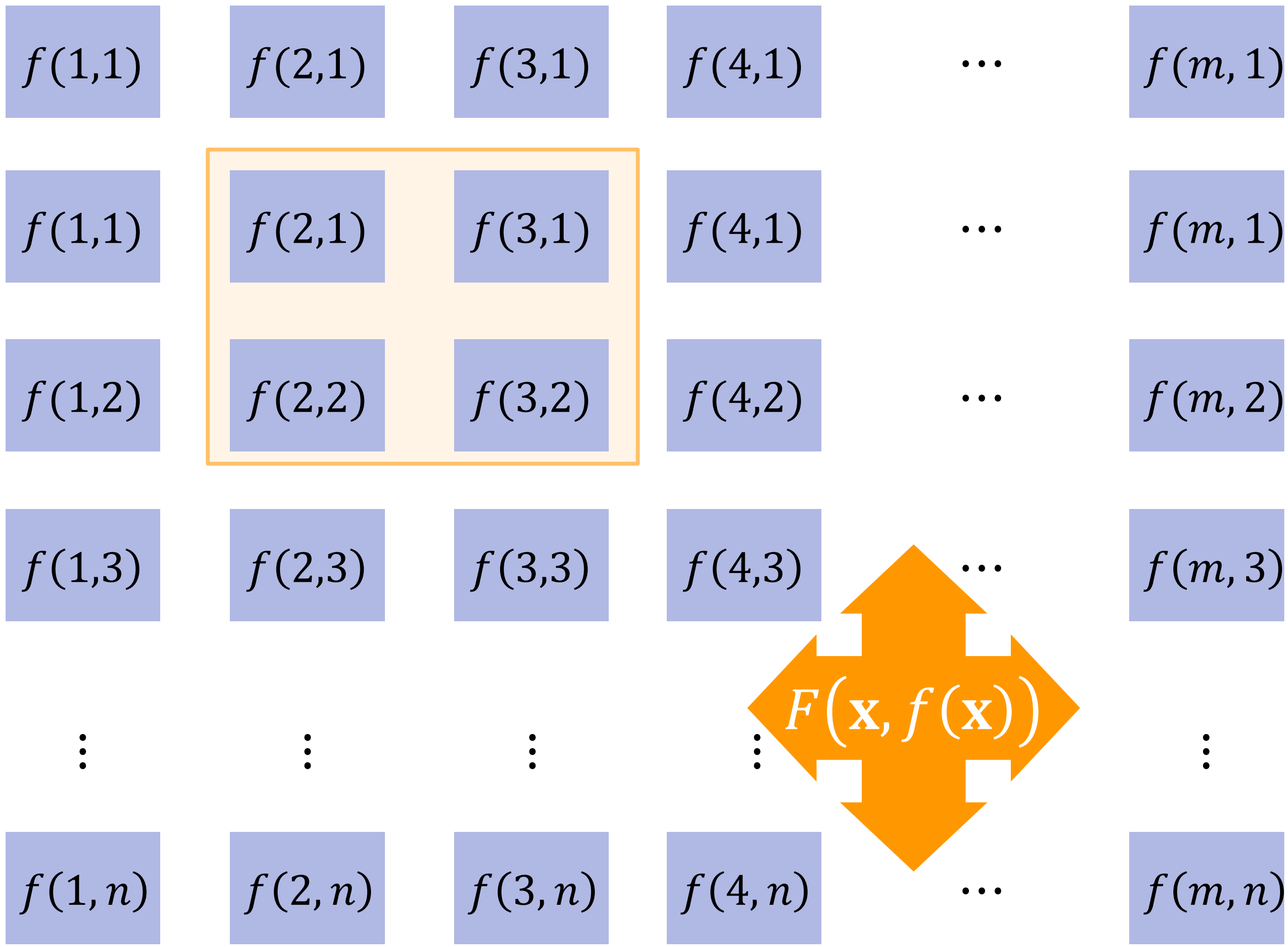
D: differential operator, including partials

Example

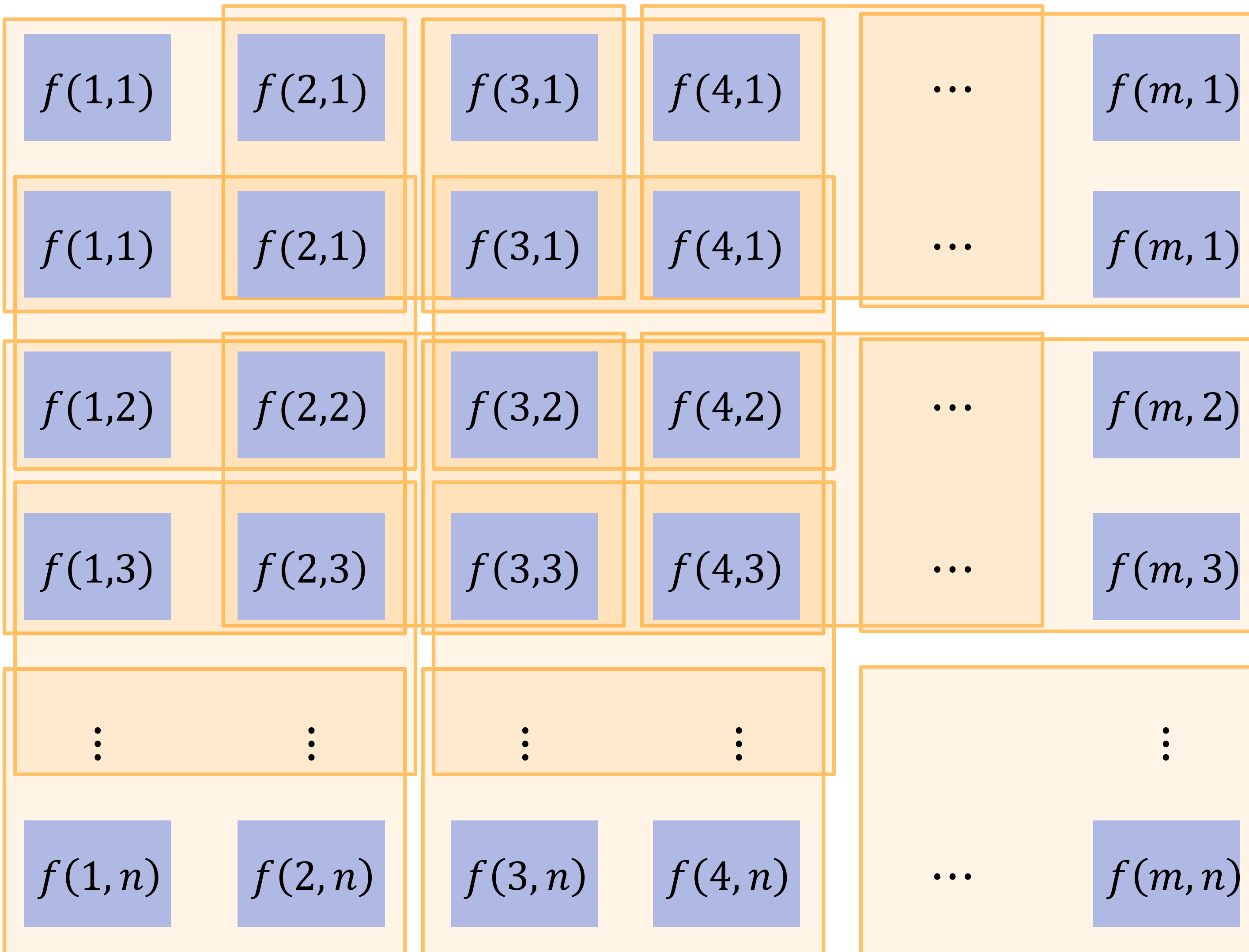
$$\frac{\partial^2}{\partial_{x_1}^2} f(\mathbf{x}) + \frac{\partial^2}{\partial_{x_2}^2} f(\mathbf{x}) = g(\mathbf{x})$$

Laplacian Δf

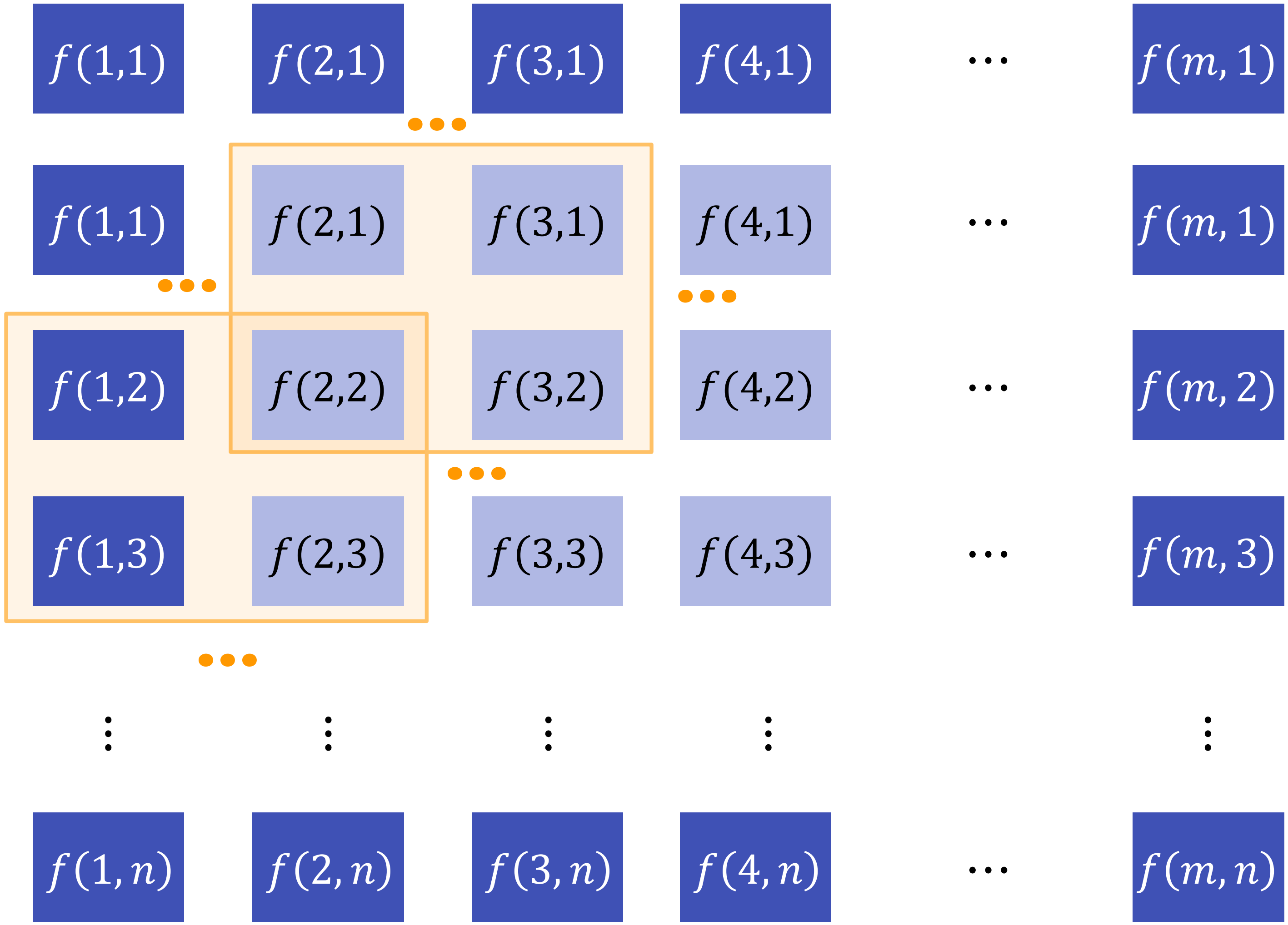
Structure of PDE



Structure of PDE



Boundary Value Problem



Solving PDEs

[Orzan et al. 2008]

[Orzan et al. 2008]

$\Delta f(x) = 0$

– and –

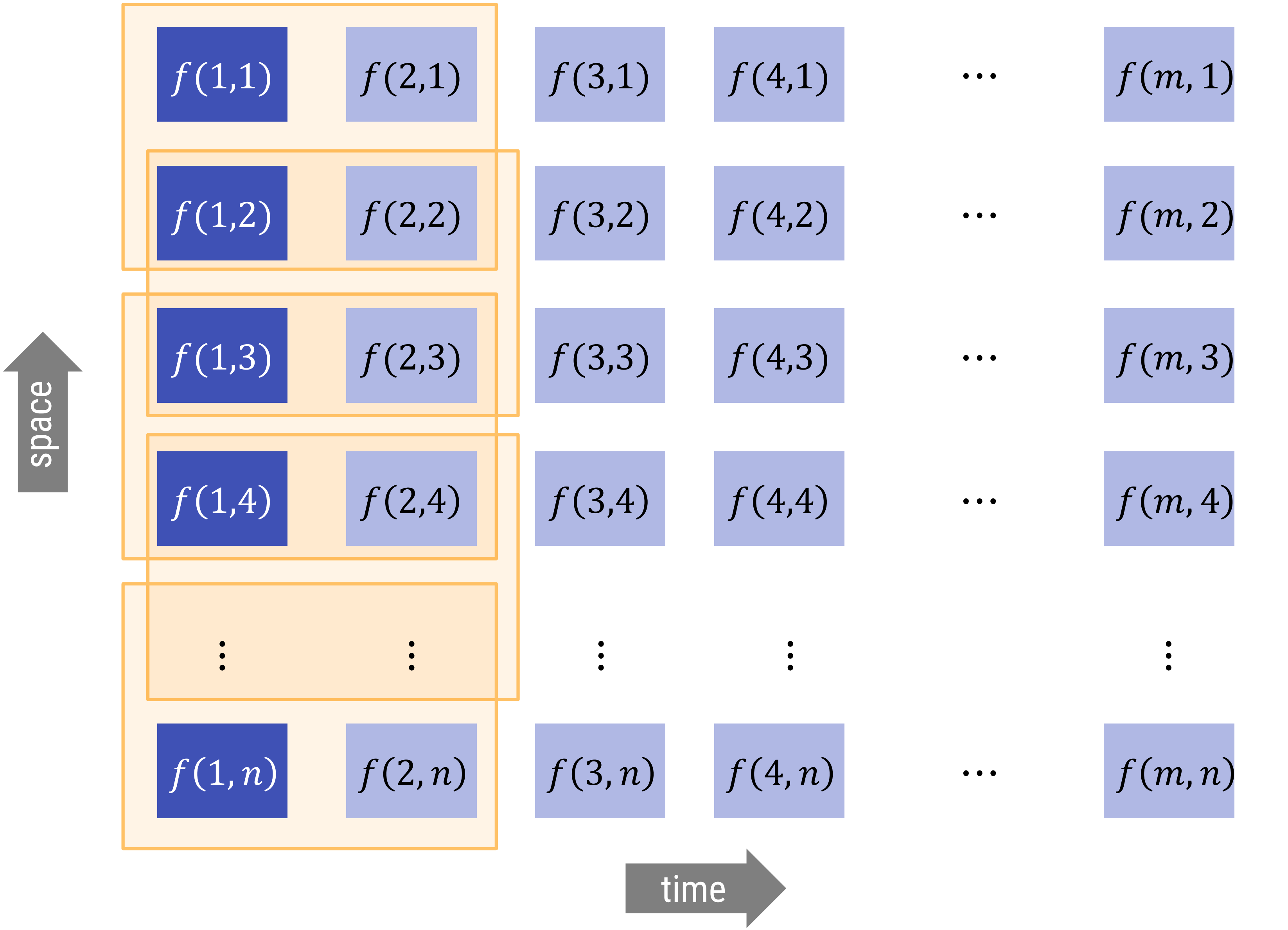
$f(x) = g(x)$ for $x \in$ curve

Usually boundary value problems

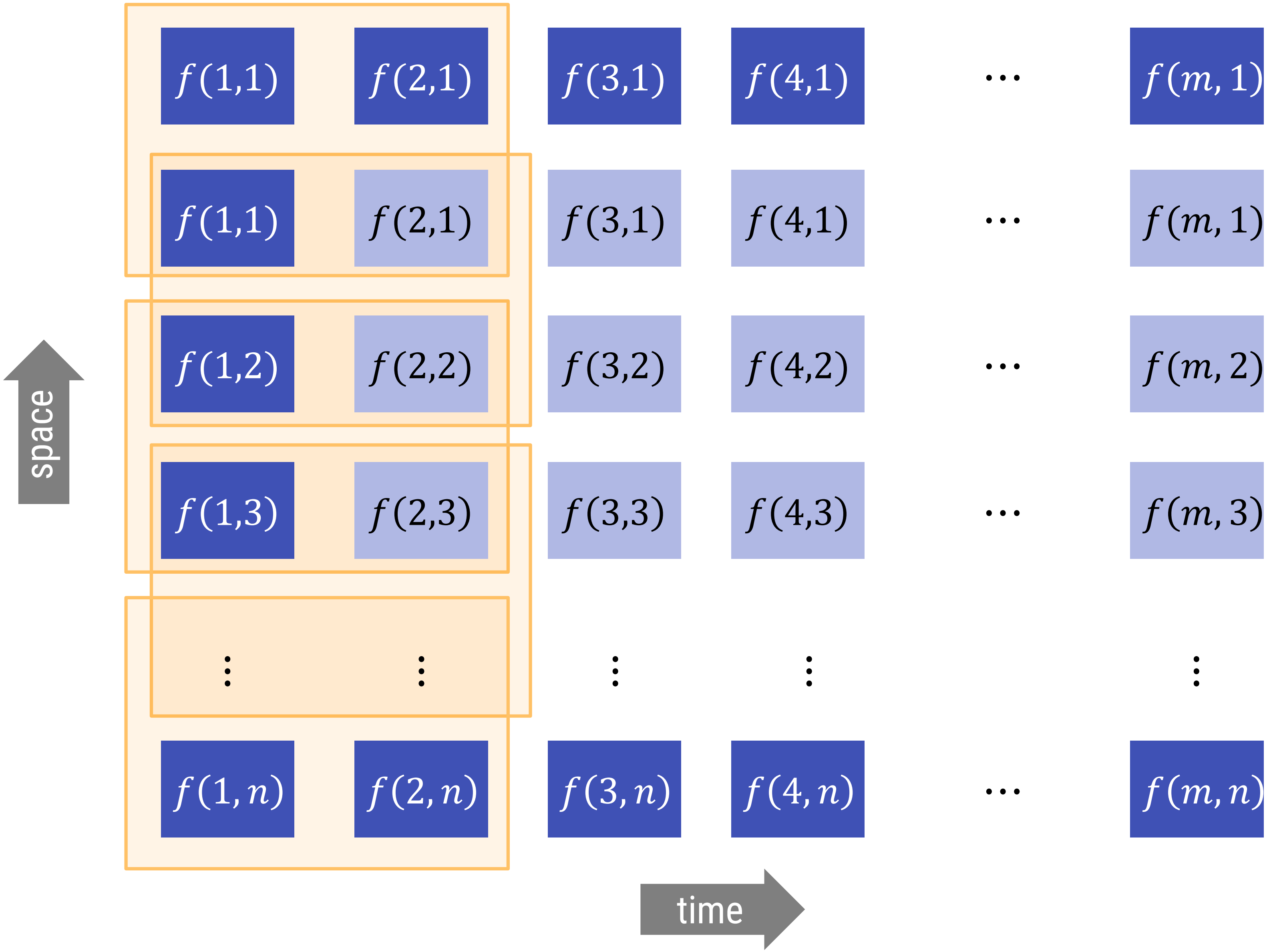
- Need global solver anyways
- No time stepping
- Linear PDEs: linear system of equations
  - We are looking at that case

A. Orzan, A. Bousseau, H. Winnemöller, P. Barla, J. Thollot, D. Sales:
Diffusion Curves: A Vector Representation for Smooth-Shaded Images.
In: *ACM Transactions on Graphics, SIGGRAPH 2008.*

Initial Value Problem

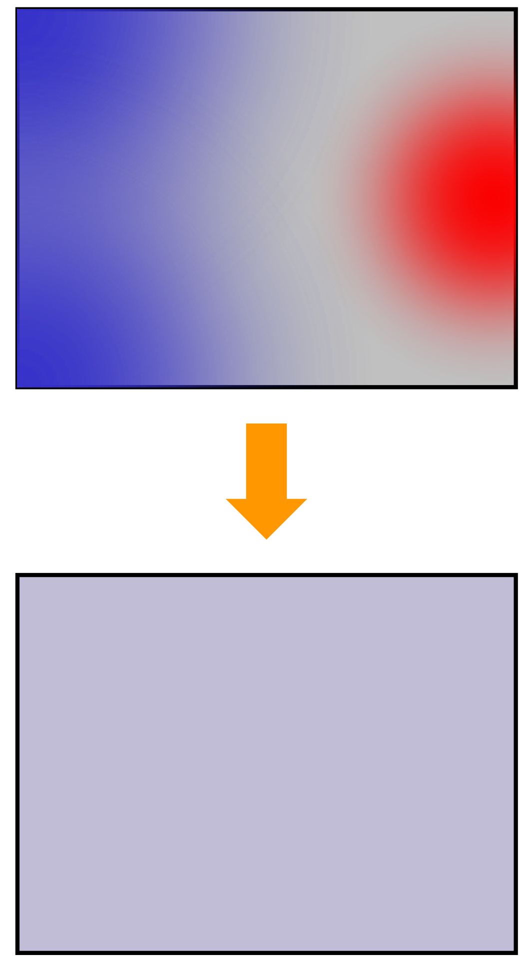


Fixed Spatial Boundaries



Examples

$$f(\mathbf{x}, 0) = g(\mathbf{x})$$



Heat Diffusion

$$f: \mathbb{R}^3 \supset (\Omega \times \mathbb{R}) \to \mathbb{R}$$

$$\partial_t f = -\lambda \underbrace{\left(\partial_x^2 + \partial_y^2\right) f}_{\Delta f}$$

Problem

- Initial heat distribution given
- Compute progression over time

Class

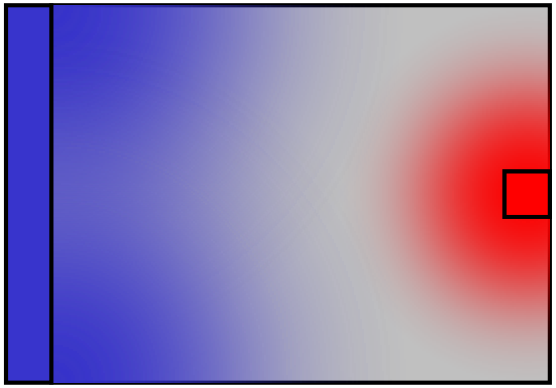
- Second order, “parabolic”
- Smoothes out details over time

Examples

Diffusion Equation

$$f: \mathbb{R}^2 \supset \Omega \to \mathbb{R}$$

$$\underbrace{\left(\partial_x^2 + \partial_y^2\right) f}_{\Delta f} = 0$$



$$\forall \mathbf{x} \in \text{Boundary}: f(\mathbf{x}) = g(\mathbf{x})$$

Problem

- Boundary conditions for heat given
- Compute steady state $\partial_t = 0$

Examples

Wave Equation

$$f : \mathbb{R}^3 \supset (\Omega \times \mathbb{R}) \rightarrow \mathbb{R}$$

$$\partial_t^2 f = \lambda \underbrace{\left(\partial_x^2 + \partial_y^2\right) f}_{\Delta f}$$

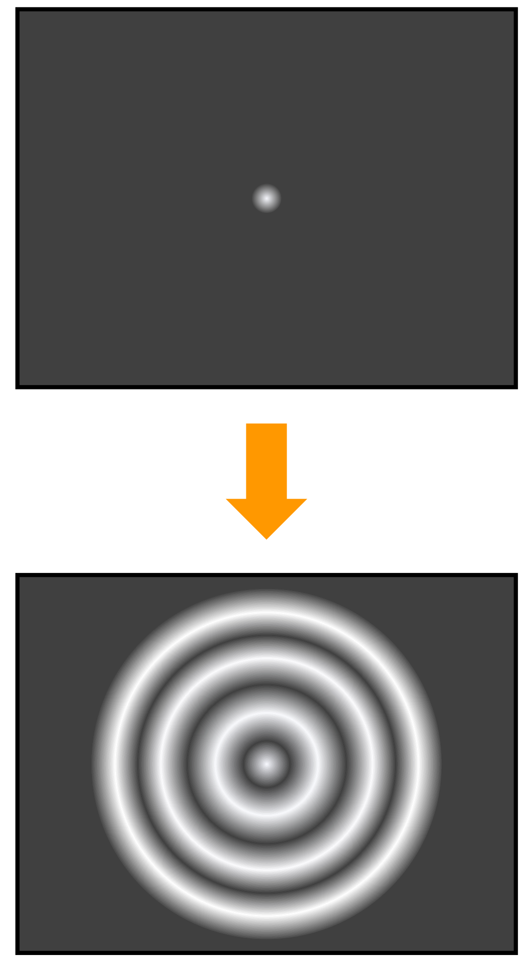
Problem

- Driver function given (time/space)
- Compute progression over time

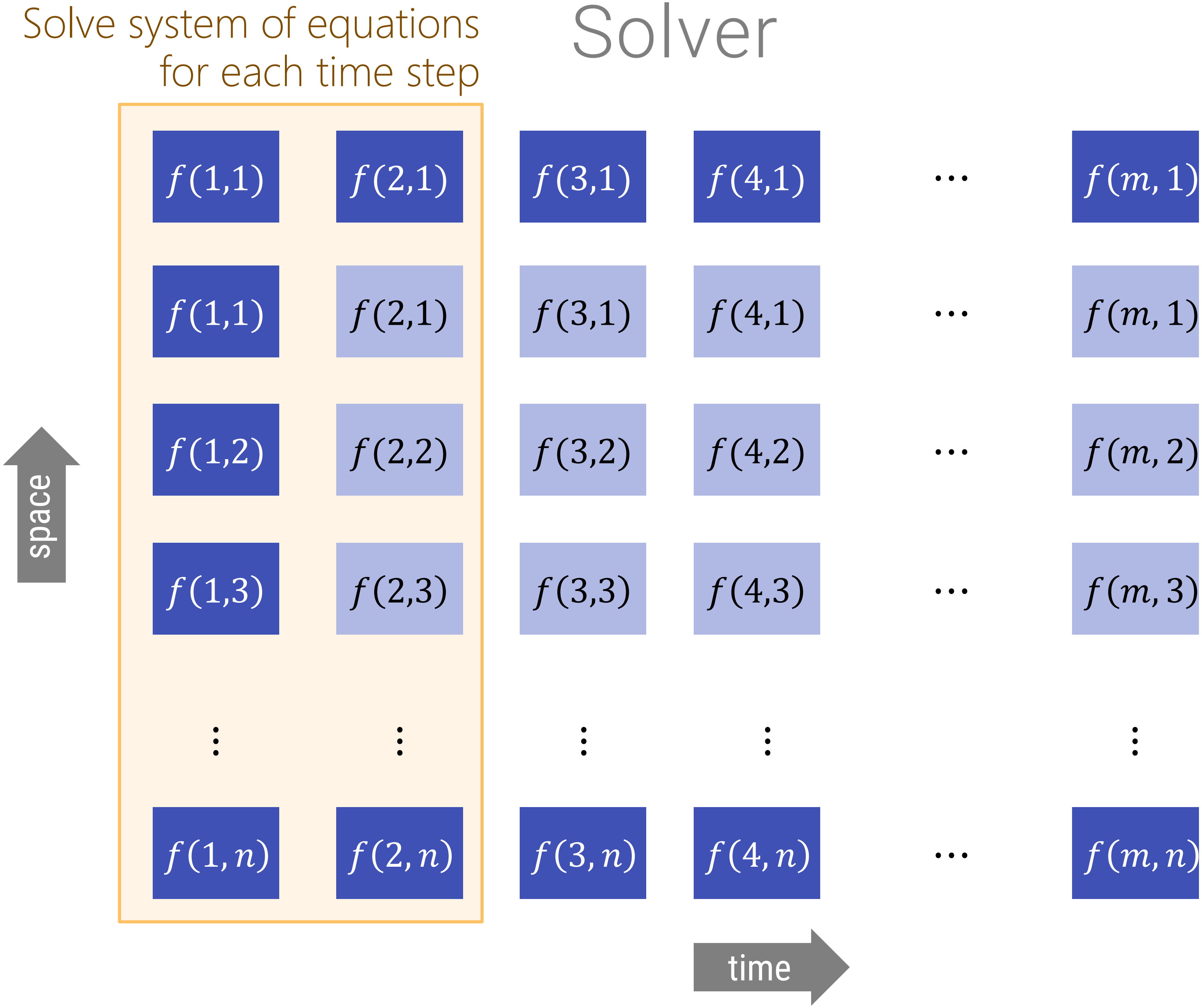
Class

- Second order, “hyperbolic”
- Transports information through space (no information loss)

$$f(\mathbf{0}, t) = g(t)$$

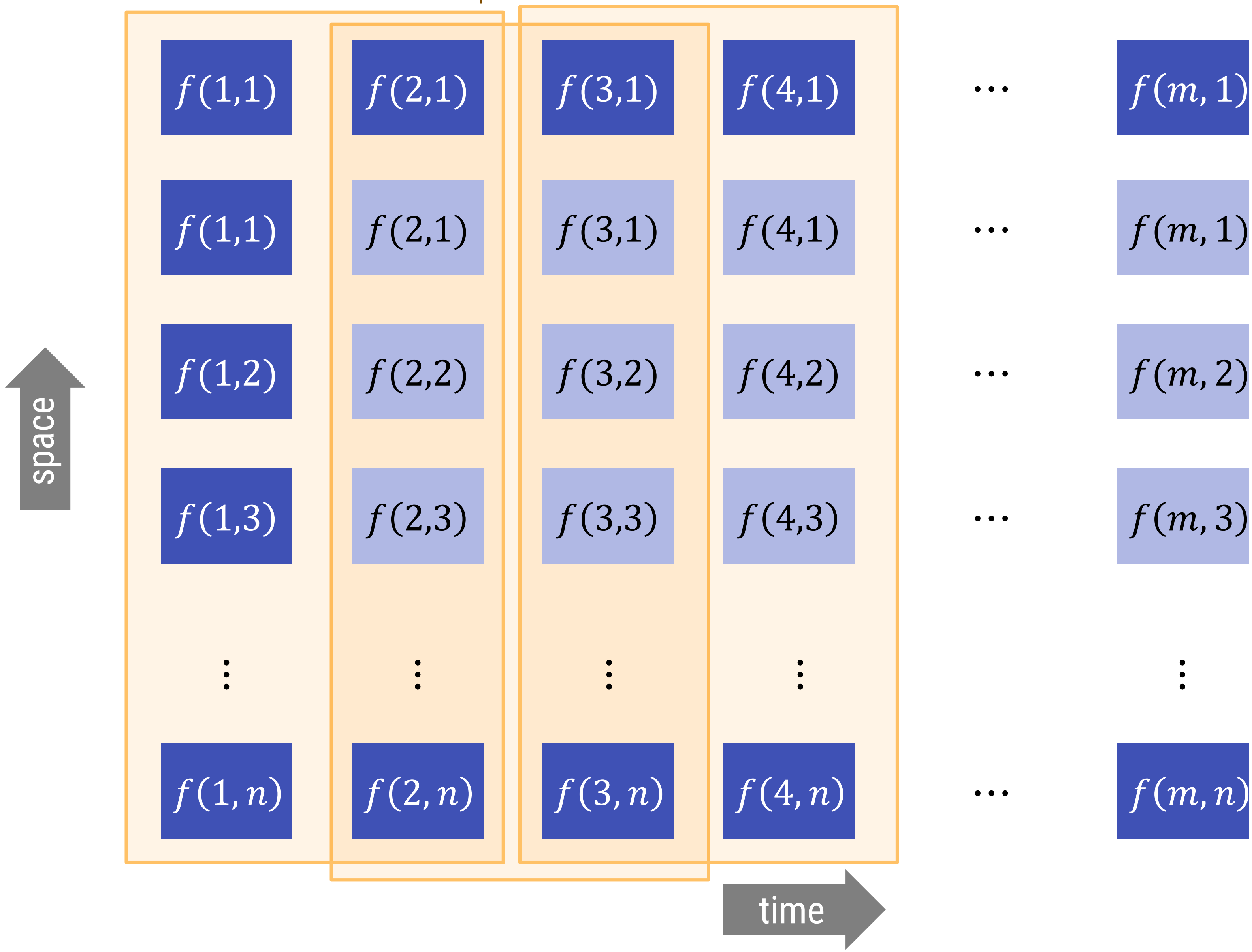


Solver


Solve system of equations
for each time step
f(1,1)
f(2,1)
f(3,1)
f(4,1)
f(m,1)
f(1,1)
f(2,1)
f(3,1)
f(4,1)
f(m,1)
f(1,2)
f(2,2)
f(3,2)
f(4,2)
f(m,2)
f(1,3)
f(2,3)
f(3,3)
f(4,3)
f(m,3)
f(1,n)
f(2,n)
f(3,n)
f(4,n)
f(m,n)
space
time

Solver

Solve system of equations
for each time step



Integral Equations

Integral Equations

Use integrals to construct L:

- For example:

$$f(x) = g(x) + \int_0^1 K(x, y) \cdot f(y) dy$$

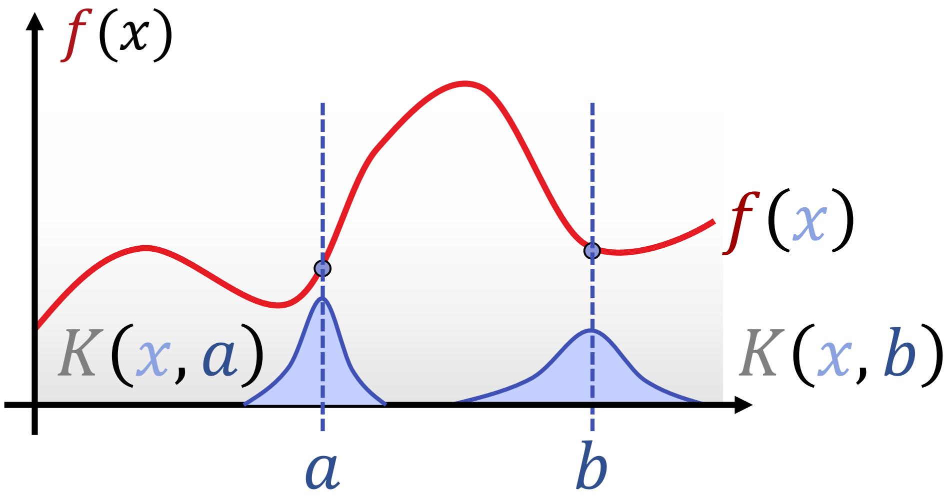
  - **Given (known):** functions K, g
  - **Unknown:** function f

- As operator equation $Lf = g$

$$L(f(x)) = \left[f(x) - \int_0^1 K(x, y) \cdot f(y) dy \right]$$

$$g = g(x)$$

What does it do?

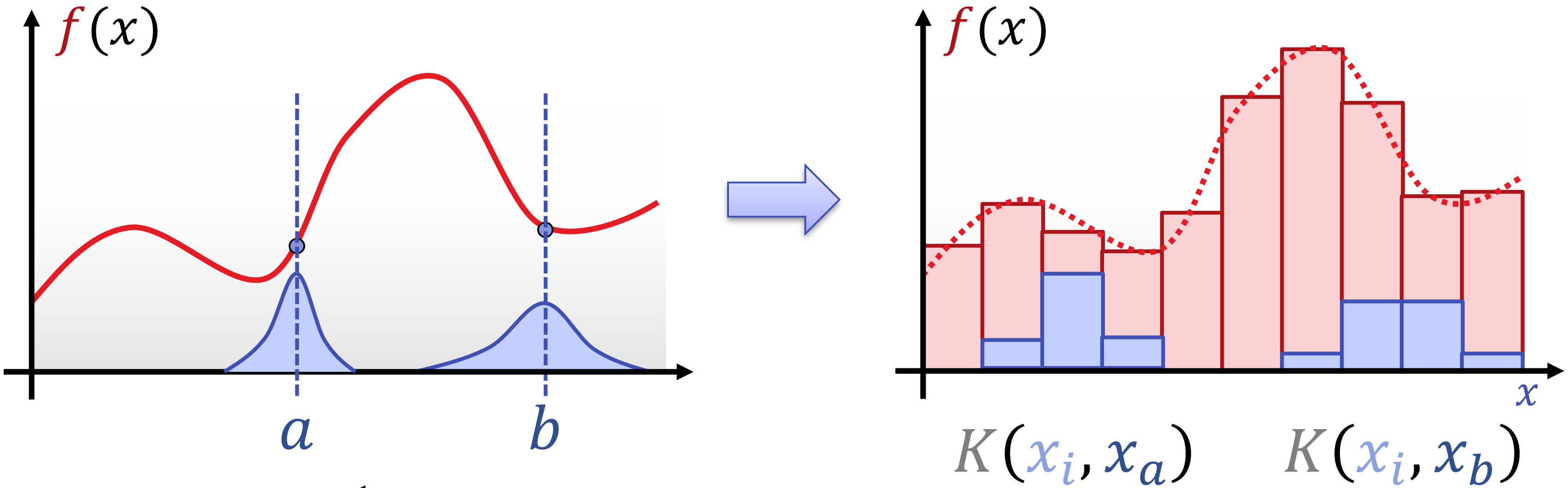


$$f(x) = g(x) + \int_0^1 K(x, y) \cdot f(y) dy$$

Fredholm integral equation (2nd kind):

- Prescribe weighted averages of function values
- Add constant function

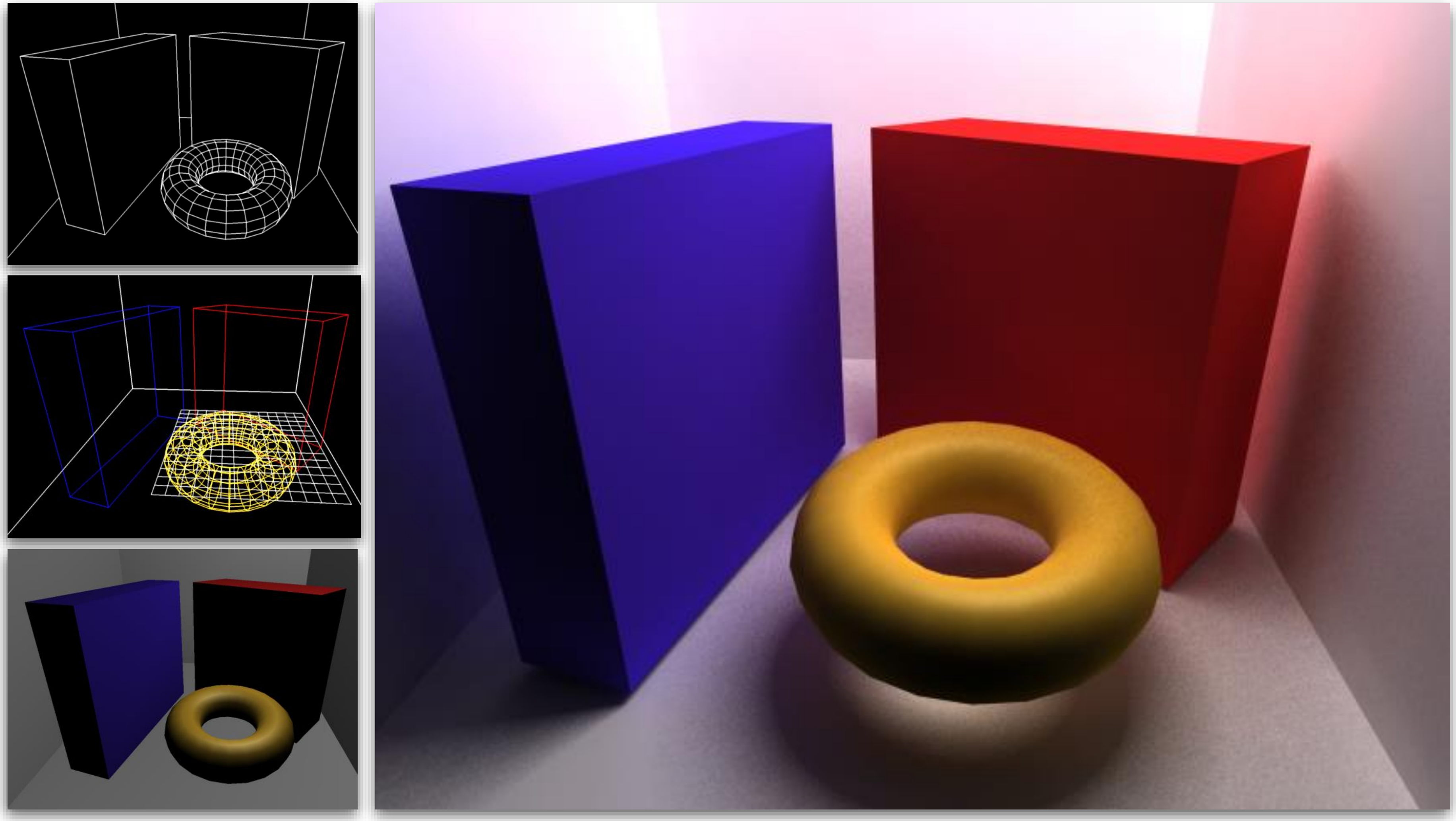
Discrete Analogy



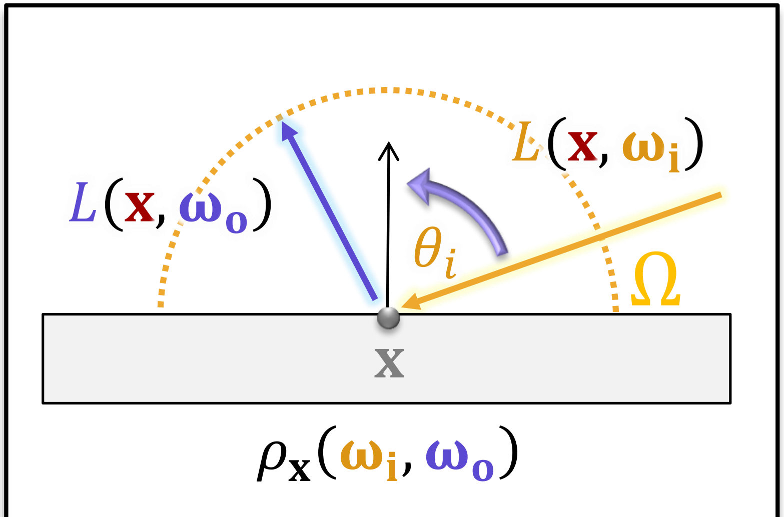
$$f(x) = g(x) + \int_0^1 K(x,y) \cdot f(y)\,dy$$

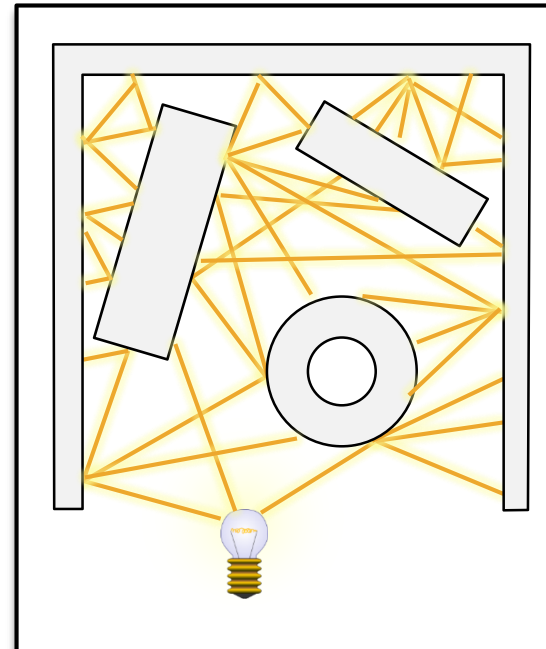
$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}, \quad \text{i.e.,} \quad \left[\mathbf{I} - \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 3 \end{pmatrix}$$

“Global Illumination”



Example: Rendering Equation



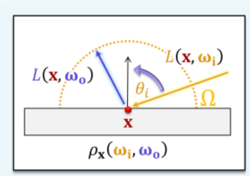


Rendering Equation

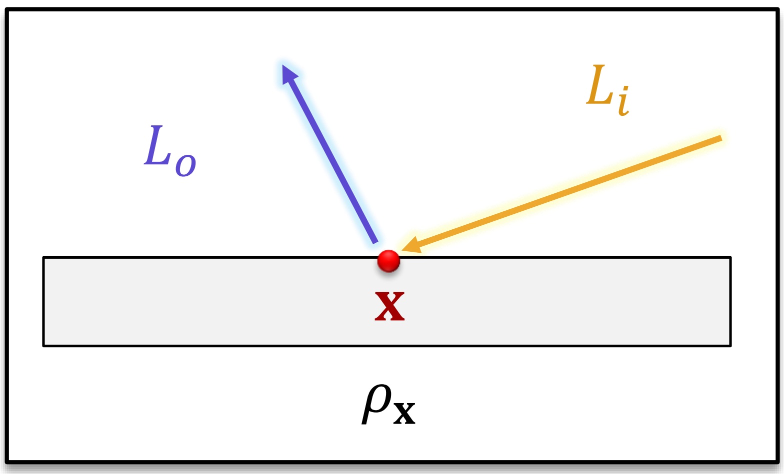
$$L(\mathbf{x}, \omega_o) = \underbrace{E(\mathbf{x}, \omega_o)}_{\text{emission}} + \underbrace{\int_{\omega_i \in \Omega} \left[L(\mathbf{x}, \omega_i) \cdot \rho_{\mathbf{x}}(\omega_i, \omega_o) \cdot \cos \theta_i \right] d\omega_i}_{\text{reflection}}$$

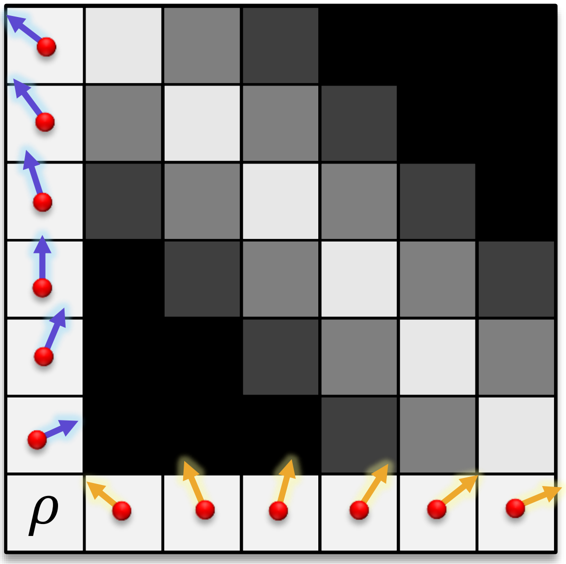
emission

reflection



Interaction with Surfaces

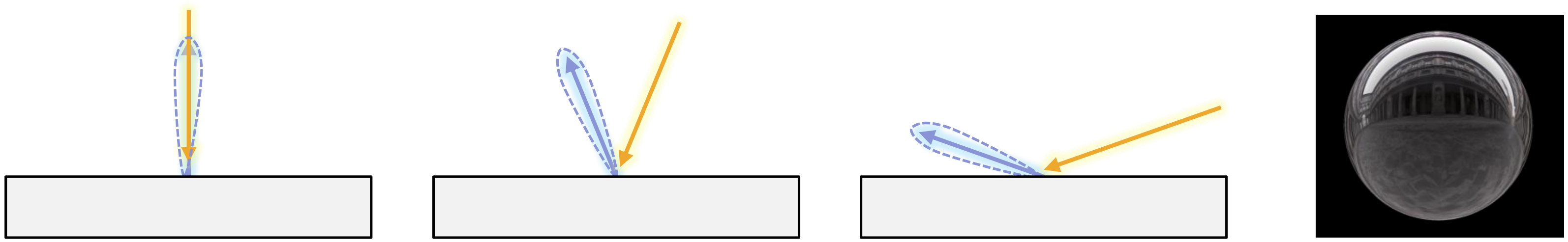




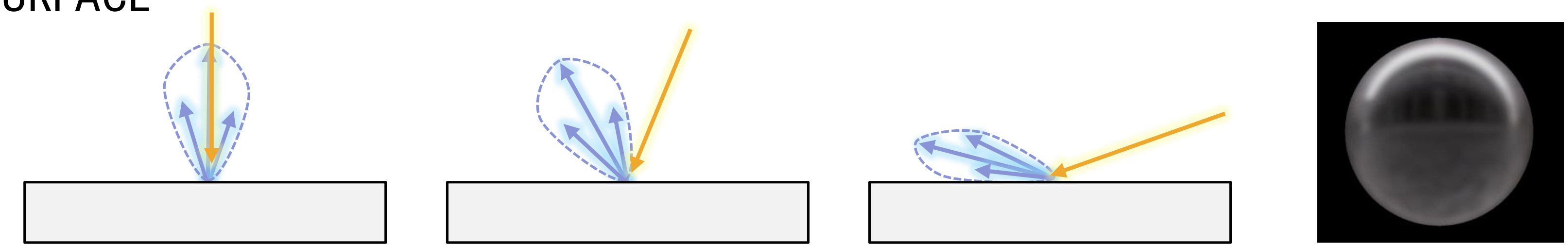
Bi-direction Reflectance Distribution Function (BRDF)

Bidirectional Reflectance Distribution Function (BRDF)

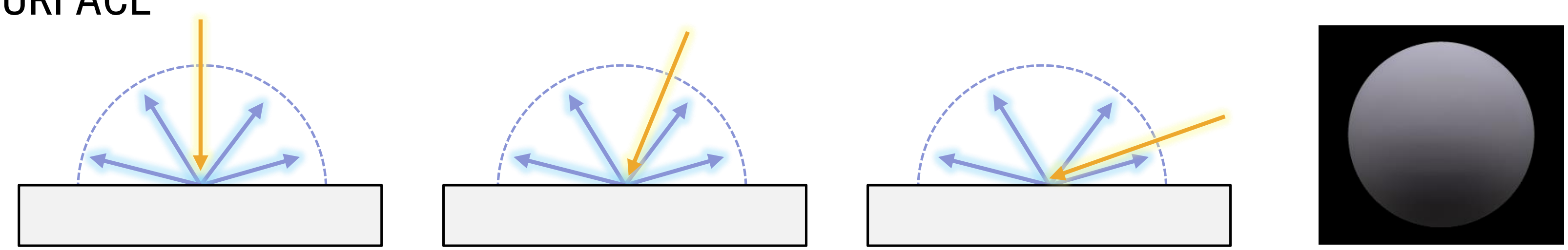
MIRROR



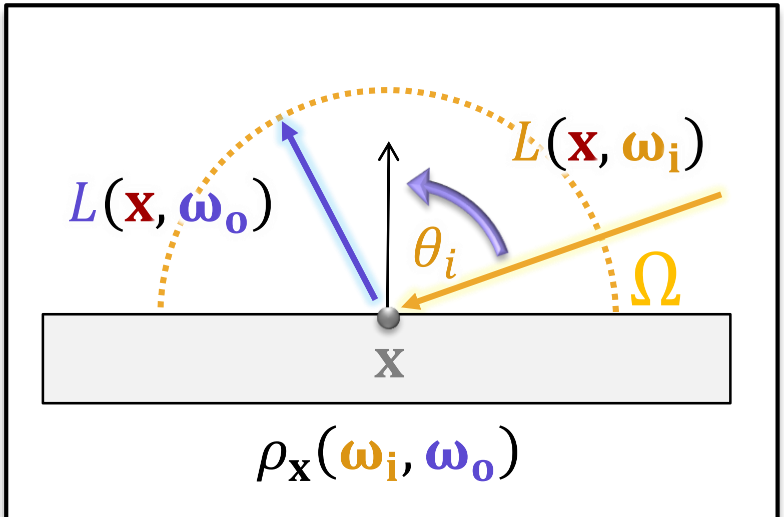
GLOSSY SURFACE

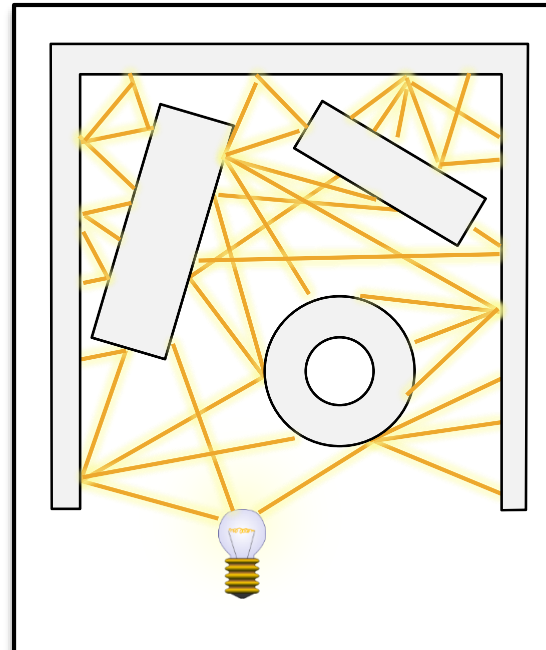


DIFFUSE SURFACE



Example: Rendering Equation



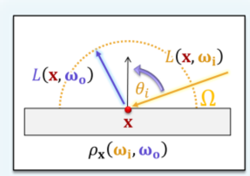


Rendering Equation

$$L(\mathbf{x}, \omega_o) = \underbrace{E(\mathbf{x}, \omega_o)}_{\text{emission}} + \underbrace{\int_{\omega_i \in \Omega} \left[L(\mathbf{x}, \omega_i) \cdot \rho_{\mathbf{x}}(\omega_i, \omega_o) \cdot \cos \theta_i \right] d\omega_i}_{\text{reflection}}$$

emission

reflection



Classification

Linear Functional Equations

Linear Functional Equation

- $Lf = g$ – solving a linear system
  - Discretization with "array"
    - "Finite differences" for differential equations
    - Replace $f'(x)$ with $[f(x_i) - f(x_{i-1})]/(x_i - x_{i-1})$
  - Discretization with linear ansatz: "finite elements"
  - Analytical solution?
    - If L is diagonizable and we know the eigenfunctions: Diagonal system
    - Scaling of projections of g on eigenfunctions

Linear Functional Equations

Linear Time Evolution

- Linear ODE ($\mathbf{x} \in \mathbb{R}^d$)
  - $\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}$
  - Solution
$$\mathbf{x}(t) = \exp(t \cdot \mathbf{A})\,\mathbf{x}(0)$$
  - $\mathbf{A}$ diagonizable?
$$\begin{aligned}&\mathbf{U}\exp(t \cdot \mathbf{D})\,\mathbf{U}^{\mathrm{T}}\mathbf{x}(0)\\ &= \mathbf{U}\begin{pmatrix}\exp(\lambda_1)^t & & \\ & \ddots & \\ & & \exp(\lambda_d)^t\end{pmatrix}\mathbf{U}^{\mathrm{T}}\mathbf{x}(0)\end{aligned}$$
  - $\mathbf{A}$ not diagonizable? – Jordan Normal Form
  - Inhomogeneous case similar*

*) see e.g. https://de.wikipedia.org/wiki/Matrixexponential

Linear Functional Equations

Linear Time Evolution

- Linear PDE with simple (Markovian) time evolution
  - $\frac{d}{dt} f = \mathrm{L} f$
  - Solution
$$\mathbf{x}(t) = \exp(t \cdot \mathrm{L})\,\mathbf{x}(0)$$
  - L diagonizable? ("self-adjoint" = symmetric?)
    - Eigenfunctions u_i

$$f(t) = \sum_i u_i(t) \cdot \exp(t \cdot \lambda_1) \langle f(0), u_i \rangle$$

Linear Functional Equations

Shift invariant Operators

- Linear Operator shift invariant?
  - (Time invariant) ordinary differential equations
  - (Spatially/temporally uniform) partial differential equations
- We know the eigenbasis already!
  - Fourier-basis
  - For example in time evolution: We can write down the solution by exponential scaling of Fourier-coefficients
  - See tutorials (heat equation)